A Note on "Extension, Spin and Non Commutativity"

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Abstract

We show that the Dirac theory of the electron, corresponds to recent approaches based on a Non commutative spacetime.

1 Introduction

As is well known, in Quantum Theory, as we go down to arbitrarily small spacetime intervals, we begin to encounter arbitrarily large momenta and energy. This in Quantum Field Theory manifests itself in the form of divergences. Indeed in the Dirac Theory of the electron [1] the position operator takes on complex eigen values and it is only after an averaging over intervals at the Compton scale that we recover usual physics. This non Hermiticity of the position operator has been recognised for a long time [2, 3], and meaningful position operators have been constructed. The crux of the matter, in this latter approach, as also in the Foldy-Wolthysen transformation, which achieves the same purpose [3, 4], is that operators are constructed in such a way that the so called positive energy spinor and negative energy spinor of the Dirac bi-spinor wave function, do not mix. This is quite meaningful because the negative energy spinors are negligible outside the Compton wavelength region. Indeed, Dirac's averaging over the Compton scale, referred to, achieves the same purpose– it eliminates zitterbewegung effects.
The spirit of Dirac's average spacetime intervals rather than spacetime points has continued to receive attention over the years in the form of minimum

spacetime intervals– from the work of Snyder and Schild to Quantum Superstring theory [5, 6, 7, 8, 9, 10, 11]. In modern language, it is symptomatic of a Non commutative spacetime geometry [12, 13, 14]. We will now highlight the intimate connection between spin, non commutativity and extension in the above context.

2 The Non commutative Structure

Indeed Newton and Wigner showed that the correct physical coordinate operator is given by

$$x^k = (1+\gamma^0)\frac{p_0^{3/2}}{(p_0+\mu)^{1/2}}\left(-\frac{\imath\partial}{\partial p_k}\right)\frac{p_0^{-1/2}}{(p_0+\mu)^{1/2}}E \tag{1}$$

where E is a projection operator given by

$$E = \frac{1}{2}p_0(E\gamma^k p_k + \mu)\gamma^0$$

and the gammas denote the usual Dirac matrices.
To appreciate the significance of (1), we first consider the case of spin zero. Then (1) goes over to

$$x^k = \imath\frac{\partial}{\partial p_k} + \frac{1}{8\pi}\int\frac{\exp(-\mu|(x-y|)}{|x-y|}\frac{\partial}{\partial y}dy \tag{2}$$

The first term on the right side of (2) denotes the usual position operator, but the second term represents an imaginary part, which has an extension $\sim 1/\mu$, the Compton wavelength, exactly as in the case of the Dirac electron. Returning to Dirac's treatment [1], the position coordinate is given by

$$\vec{x} = \frac{c^2pt}{H} + \frac{1}{2}\imath c\hbar(\vec{\alpha} - c\vec{p}H^{-1})H^{-1} \equiv \frac{c^2p}{H}t + \hat{x} \tag{3}$$

H being the Hamiltonian operator and α's the non-commuting Dirac matrices, given by

$$\vec{\alpha} = \left[\begin{array}{cc} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{array}\right]$$

The first term on the right hand side is the usual (Hermitian) position. The second term of $\vec{x}$ is the small oscillatory term of the order of the Compton wavelength, arising out of zitterbewegung effects which averages out to zero. On the other hand, if we were to work with the (non Hermitian) position operator in (3), then we can easily verify that the following Non-commutative geometry holds,

$$[x_\imath, x_j] = \alpha_{\imath j} l^2 \tag{4}$$

where $\alpha_{\imath j} \sim 0(1)$.
The relation (4) shows on comparison with the position-momentum commutator that the coordinate $\vec{x}$ also behaves like a "momentum". This can be seen directly from the Dirac theory itself where we have

$$c\vec{\alpha} = \frac{c^2 \vec{p}}{H} - \frac{2\imath}{\hbar}\hat{x}H \tag{5}$$

In (5), the first term is the usual momentum. The second term is the extra "momentum" $\vec{\hat{p}}$ due to the relations (4).
Infact we can easily verify from (5) that

$$\vec{\hat{p}} = \frac{H^2}{\hbar c^2}\hat{x} \tag{6}$$

where $\hat{x}$ has been defined in (3).
We finally investigate what the angular momentum $\sim \vec{x} \times \vec{p}$ gives - that is, the angular momentum at the Compton scale. Using (3), we can easily show that

$$(\vec{x} \times \vec{p})_z = \frac{c}{E}(\vec{\alpha} \times \vec{p})_z = \frac{c}{E}(p_2 \alpha_1 - p_1 \alpha_2) \tag{7}$$

where E is the eigen value of the Hamiltonian operator H. (7) shows that the angular momentum leads to the "mysterious" Quantum Mechanical spin.
In the above considerations, we started with the Dirac equation and deduced the underlying Noncommutative geometry of spacetime. Interestingly, starting with Snyder's Non commutative geometry, based solely on Lorentz invariance and a minimum spacetime length, at the Compton scale,

$$[x, y] = \frac{\imath l^2}{\hbar} L_z etc.$$

that is, in effect starting with (4), it is possible to deduce the relations (7),(6) and the Dirac equation itself [10, 15, 16, 17].

We have thus established the correspondence between considerations starting from the Dirac theory of the electron and Snyder's (and subsequent) approaches based on a minimum spacetime interval and Lorentz covariance.

3 Concluding Remarks

We remark that in the usual Quantum Field Theory we encounter divergences which require renormalization. The motivation for the extension of particles, or the Non commutative structure (4) has been to circumvent these divergences. In this sense the Non commutative geometry represents "renormalised" coordinates.

We consider in a little more detail[17] the implications of Dirac's averaging over the Compton scale.

We consider for simplicity, the free particle Dirac equation. The solutions are of the type,

$$\psi = \psi_A + \psi_S \tag{8}$$

where

$$\psi_A = e^{\frac{\imath}{\hbar}Et} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \text{ or } e^{\frac{\imath}{\hbar}Et} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \text{ and}$$

$$\tag{9}$$

$$\psi_S = e^{-\frac{\imath}{\hbar}Et} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ or } e^{-\frac{\imath}{\hbar}Et} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

denote respectively the negative energy and positive energy solutions. From (8) the probability of finding the particle in a small volume about a given point is given by

$$|\psi_A + \psi_S|^2 = |\psi_A|^2 + |\psi_S|^2 + (\psi_A \psi_S^* + \psi_S \psi_A^*) \tag{10}$$

Equations (9) and (10) show that the negative energy and positive energy solutions form a coherent Hilbert space and so the possibility of transition to negative energy states exists. This difficulty however can be overcome by the well known Hole theory which uses the Pauli exclusion principle, and is described in many standard books on Quantum Mechanics.

However the last or interference term on the right side of (10) is like the zitterbewegung term. When we remember that we really have to consider averages over space time intervals of the order of $\hbar/mc$ and $\hbar/mc^2$, this term disappears and effectively the negative energy solutions and positive energy solutions stand decoupled in what is now the physical universe.

A more precise way of looking at this is[3] that as is well known, for the homogeneous Lorentz group, $\frac{p_0}{|p_0|}$ commutes with all operators and yet it is not a multiple of the identity as one would expect according to Schur's lemma: The operator has the eigen values ± 1 corresponding to positive and negative energy solutions. This is a super selection principle [18] pointing to the two incoherent Hilbert spaces or universes now represented by states ψ_A and ψ_S which have been decoupled owing to the averaging over the Compton scale spacetime intervals. But the energies we usually consider are such that our length scale is much greater than the Compton wavelength– as if we were in the usual poiint spacetime manifold.

Thus once again we see that outside the Compton scale region we recover the usual physics.

It may be mentioned that in Quantum Superstring theory also we encounter Non commutative relations like (3)[19].

It may also be mentioned that Zakruzewski has shown from a classical viewpoint that spin implies Non commutative spacetime[20].

We finally make the following comment:

It can easily be verified that if we specialise to the one dimensional case, then the Dirac coordinate (3) becomes identical to the Newton-Wigner coordinate, which latter however defines a commutative spacetime in three dimensions (Cf.ref.[3]). Further the Nelson-Nottale approach which uses a double Weiner stochastic process leads to a complex coordinate on the one hand, and the Schrodinger equation on the other (Cf.ref.[17] and [21]). Such a complex coordinate also appears in the de Broglie-Bohm approach (Cf.ref.[17] for details). On the other hand in the Dirac coordinate (3), this additional coordinate is directly related to zitterbewegung effects. We can see that the double Weiner process alluded is also connected with such an effect: The negative time derivative, which does not equal the positive time derivative in this case, represents negative energy states.

However if we analyse the stochastic Schrodinger equation question further, and generalise the one dimensional case we consider to three dimensions, then as is well known[22],

$$\imath \to \vec{\sigma},$$

where $\vec{\sigma}$ are the Pauli matrices. In other words, we not only cross over to special relativity, but also recover the non commutative geometry (4) or Snyder's relations alluded to, at the Compton scale. Thence it is possible to derive the Dirac equation itself (Cf.ref.[17]).
In other words non commutative spacetime and spin can be shown to originate from stochastic double Weiner processes at the micro level.

References

[1] P.A.M. Dirac, "The Principles of Quantum Mechanics", Clarendon Press, Oxford, 1958, pp.4ff, pp.253ff.

[2] T.D. Newton, and E.P. Wigner, Rev. Mod. Phys., 21 (3), 1949, pp.400ff.

[3] S.S. Schweber, "Relativistic Quantum Field Theory", Harper and Row, New York, 1964, p.47.

[4] J.D. Bjorken and S.D. Drell, "Relativisitic Quantum Mechanics", McGraw Hill, New York, 1964, p.24.

[5] A. Schild, Phys.Rev., 73, 1948, p.414-415.

[6] H.S. Snyder, Physical Review, Vol.72, No.1, July 1 1947, p.68-71.

[7] V.G. Kadyshevskii, J.Exptl.Theoret.Phys., USSR, 41, December 1961, p.1885-1894.

[8] C. Wolf, Hadronic Journal, Vol.13, 1990, p.22-29.

[9] T.D. Lee, "Particle Physics and Introduction to Field Theory", Harwood Academic, 1981, pp.383ff.

[10] B.G. Sidharth, Chaos, Solitons and Fractals, 11(8), 2000, 1269-1278.

[11] B.G. Sidharth, Chaos, Solitons and Fractals, 13(2), 2002, p.189-193.

[12] J. Madore, Class.Quantum Grav. 9, 1992, p.69-87.

[13] B.G. Sidharth, "A Reconciliation of Electromagnetism and Gravitation", to appear in Annales de la Fondation Louis d Broglie.

[14] B.G. Sidharth, Il Nuovo Cimento, 116B (6), 2001, pg.4 ff.

[15] B.G. Sidharth, Frontiers of Fundamental Physics 4, Plenum Publishers/Kluwer Academic, New York, 2001, p.97-107.

[16] B.G. Sidharth, "Spin and Non Commutative Geometry", to appear in Chaos, Solitons and Fractals.

[17] B.G. Sidharth, "Chaotic Universe: From the Planck to the Hubble Scale", Nova Science Publishers, New York, 2001, p.20.

[18] P. Roman, "Advanced Quantum Theory", Addison-Wesley, Reading, Mass, 1965, p.31.

[19] Y. Ne'eman, in Proceedings of the First Internatioinal Symposium, "Frontiers of Fundmental Physics", Eds. B.G. Sidharth and A. Burinskii, Universities Press, Hyderabad, 1999, p.83-96.

[20] S. Zakruzewski in "Quantization, Coherent States and Complex Structures", Eds. J.P. Antoine et al., Plenum Press, New York, 1995, pp.249-255.

[21] L. Nottale, "Fractal Space-Time and Microphysics: Towards a Theory of Scale Relativity", World Scientific, Singapore, 1993, p.145ff.

[22] M. Sachs, "General Relativity and Matter", D. Reidel Publishing Company, Holland, 1982, p.45ff.